A CONSTRUCTION OF A SKEWAFFINE STRUCTURE IN LAGUERRE GEOMETRY

ANDRZEJ MATRAŚ

Abstract. J. Andre constructed a skewaffine structure as a group space of a normally transitive group. In the paper this construction was used to describe such an external structure associated with a point of Laguerre plane. Necessary conditions for ensure that the external structure is a skewaffine plane are given.

Introduction

The internal incidence structure associated with a point p of a Benz plane consists of all points nonparallel to p and, as lines, all circles passing through p (extended by all parallel classes not passing through p). This is an affine plane, called the derived affine plane in p. Natural question arises about a characterization of the external structure in p by some linear structure.
Among a wide class of noncommutative (in general) linear structures constructed by J. Andre (cf. [1]) there are skewaffine planes which are good candidates to obtain the characterization we are looking for. One of classical example of a skewaffine plane is the set of circles of Euclidean plane with centers of circles as basepoints (cf. [1]). Under the weak conditions Wilbrink (cf. [11]) constructed in a fixed point of Minkowski plane known as residual nearaffine plane - a skewaffine plane such that any straight line intersects any nonparallel line in exactly one point. In the examples (circles and hyperbolas) the construction is based on the observation that any circle (resp. hyperbola) has exactly one center which can be taken as a basepoint of a line corresponding to a given conic. This center is the image of our point p in the inversion with respect to the circle. In the case of Laguerre planes this construction cannot be used, since the image of the point p in the inversion is parallel to p and the inversion does not distinguished any point which can be taken as a base point. It seems to be natural to investigate a symmetry with two pointwiese fixed generators. Two fixed generators do not define a symmetry of Laguerre plane. To make the construction uniquely determinated we point out an invariant pencil $\langle p, K\rangle$ of circles tangent in the point p. A basepoint of a line corresponding to a circle (which does not pass through p) we obtain as a point of tangency of the circle with some circle of the pencil $\langle p, K\rangle$. Opposite to Möbius and Minkowski planes basepoints belong to corresponding circles and the residual skewaffine plane is not determined by the point p only, but by the pencil $\langle p, K\rangle$.
A large class of regular skewaffine planes was given in [1], [9] as the group space $\mathrm{V}(\mathbf{G})$ of a normally transitive group $\mathbf{G}$. In such skewaffine planes lines are obtained as a join of a basepoint and the orbit of an other point with respect to the stabilizer of the basepoint.

1991 *Mathematics Subject Classification.* 51B20(2000).
Key words and phrases. miguelian Laguerre plane, symmetry axiom.

The starting point of our paper is a group $\mathbf{G}$ of automorphisms of Laguerre plane such that $\mathrm{V}(\mathbf{G})$ contains a set of circle not passing through p as lines. The group $\mathbf{G}$ fixes points parallel to p and the pencil $\langle p, K\rangle$. Minimal conditions for the group $\mathbf{G}$ are transitivity (cf. A1) and circular transitivity for some circle of the pencil $\langle p, K\rangle$ (cf. A2). The latter axiom A3 stands for an assumption that for any $L \in \mathcal{C}$ with $p \notin L$ there exists exactly one $M \in \langle p, K\rangle$ tangent to L. It guarantees that each circle not passing through p corresponds to some line of $\mathrm{V}(\mathbf{G})$. We show that under the axioms A1, A2, A3 the group $\mathbf{G}$ contains of $\mathbb{L}$-translations and $\mathbb{L}$-strains fixing the pencil $\langle p, K\rangle$. This group is a subdirect product of the normal subgroup of $\mathbb{L}$-translations and an arbitrary subgroup of $\mathbb{L}$-strains with a fixed center (cf. Theorem 3.2). We point out an example of nonovoidal Laguerre plane which satisfies our assumptions for some pencil $\langle p, K\rangle$ (cf. Remark 2.2). Miquelian Laguerre planes of characteristic distinct from 2 fulfils the axioms for any pencil (cf. Remark 2.1).
We use skewaffine planes to get a characterization of the tangency in Laguerre planes. We obtain the condition deciding whether a circle through two points tangent to a circle can be constructed (cf. Theorem 4.2). As an application we also show that the tangency points of circles of pencils $\langle p, K\rangle$ and $\langle q, x\rangle$, where q is parallel to p, form a circle cf. Theorem 4.1.

1. Notations and basic definitions.

A *Laguerre plane* is a structure $\mathbb{L} = (\mathcal{P}, \mathcal{C}, -)$, where

— $\mathcal{P}$ is a set of *points* denoted by small Latin letters,
— $\mathcal{C} \subset 2^{\mathcal{P}}$ is a set of *circles* denoted by capital Latin letters,
— $-$ is an equivalence relation on $\mathcal{P}$ called *parallelity* (cf., for example, [4]).

We use the notation $-$ to avoid misunderstanding with the parallelity for skewaffine planes.

The equivalence classes of the relation $-$ we call *generators* and denote by capital Latin letters. The following condition must be satisfied:

(1) Three pairwise nonparallel points can be uniquely joined by a circle.
(2) For every circle K and any two nonparallel points $p \in K, q \notin K$ there is precisely one circle L passing through q which touches K in p (i.e. $K \cap L = \{p\}$).
(3) For any point p and a circle K there exists exactly one point q such that $p \parallel q$ and $q \in K$.
(4) There exists a circle containing at least three points, but not all points.

The unique generator containing a point p we denote by $\overline{p}$. If a, b, c are points pairwise nonparallel the unique circle containing points a, b, c we denote by $(a, b, c)^{\circ}$. The circle tangent to a circle K and passing through points p, q ($p \in K, q \notin K, p \not- q$) we denote by $(p, K, q)^{\circ}$. If $p \in K$ by the symbol $\langle p, K\rangle$ we denote the pencil of circles tangent to K in a point p. If x, y are nonparallel, then the set of circles containing points x, y we call *pencil of circles with vertexes* x, y and denote by $\langle x, y\rangle$. For a point x and a circle K the unique point of K parallel to x (exists by the condition (3)) we denote by xK
The *derived plane in a point* p of Laguerre plane $\mathbb{L}$ consist of all points not parallel to p and, as lines, all circles passing through p (excluding p) and all generators not passing through p (notation $\mathbb{A}_p$). This is an affine plane.

An automorphism of Laguerre plane is a permutation of the point set which maps circles to circles (and generators to generators). The automorphism ϕ is called *central* if there exists a fixed point p such that ϕ induces a central collineation of $\overline{\mathbb{A}_p}$ - the projective extension of the derived affine plane $\mathbb{A}_p$.
$\mathbb{L}$-*translation* is a central automorphism of Laguerre plane $\mathbb{L}$ which fixes points of $\overline{p}$ and induces a translation of $\mathbb{A}_p$ for some point p. The group of translations which fixes circles of a pencil $\langle p, K\rangle$ (respectively the familly of generators $\mathcal{G}$) is denoted by $\mathbf{T}(\overline{p}, K)$ (resp. $\mathbf{T}(\overline{p}, \mathcal{G})$) (cf. [6]).
$\mathbb{L}$-*strain* with respect to a generator is a central automorphism of $\mathbb{L}$ which fixes points of some generator $\overline{p}$ and circles of some pencil $\langle q, M\rangle$ $(p \not\parallel q)$. The group of all $\mathbb{L}$-strains fixing the points of $\overline{p}$ and the circles of the pencil $\langle q, M\rangle$ we denote by $\Delta(\overline{p}, q, M)$. The involutory automorphism which fixes pointwise generators X, Y and a circle M (not pointwise) we call *Laguerre symmetry* and denote by the symbol $\mathrm{S}_{X,Y;M}$ ($\mathrm{S}_{X,Y;M} \in \Delta(\overline{p}, q, M)$ for $p \in X$, $q \in Y$).
A group of central automorphisms is called *circular transitive* if the extension to $\overline{\mathbb{A}_p}$ of its restriction to $\mathbb{A}_p$ is linear transitive i.e. is transitive on any line passing through the center. The group of automorphisms of $\mathbb{L}$ is called $\langle p, K\rangle$-*transitive* (resp. $\overline{p}$-transitive) if it contains a circular transitive group $\mathbf{T}(\overline{p}, K)$ (resp. $\mathbf{T}(\overline{p}, \mathcal{G})$, cf. [8]). The group of automorphisms of $\mathbb{L}$ is called $(\overline{p}, \langle q, M\rangle)$-*transitive* if it contains a circular transitive group $\Delta(\overline{p}, q, M)$.
A *skewaffine space* (cf. [1]) is an incidence structure $\mathbb{S} = (X, \sqcup, \|)$, where X is a nonempty set of points denoted by small Latin letters, and

$$\sqcup : \{(x, y) \in X^2 | x \neq y\} \to 2^X$$

is a function. The sets of the form $x \sqcup y$ $(x \neq y)$ are called lines. They will be also denoted by capital Latin letters. The symbol $\|$ denotes an equivalence relation among the lines. The following axioms must be satisfied:

- (L1) $x, y \in x \sqcup y$,
- (L2) $z \in x \sqcup y \setminus \{x\}$, implies $x \sqcup y = x \sqcup z$ (exchange condition),
- (P1) given any line L and any point x there exists exactly one line $x \sqcup y$ parallel to L (Euclidean axiom),
- (P2) $\forall x, x', y, y' (x \neq y, x' \neq y' \wedge x \sqcup y \parallel x' \sqcup y' \to y \sqcup x \parallel y' \sqcup x'$ (symmetry condition),
- (T) if x, y, z are pairwiese different points such that $x \sqcup y \parallel x' \sqcup y'$ there exists a points z' such that $x \sqcup z \parallel x' \sqcup z'$ and $y \sqcup z \parallel y' \sqcup z'$ (Tamaschke's condition).

If we assume $x = x'$ in axiom (T) the condition is called affine Veblen-condition (V). We will consider additional conditions for skewaffine space:

- (Pgm) $\forall x, y, z \in X, \{x, y, x\}_{\neq} \exists w \in X$ with $x \sqcup y \parallel z \sqcup w$ and $x \sqcup z \parallel y \sqcup w$
- (Des) $\forall u, x, y, z \in X, \{u, x, y, z\}_{\neq} x' \in u \sqcup x \setminus \{u\} \to \exists y' \in u \sqcup y \setminus \{u\}, z' \in u \sqcup z \setminus \{u\}$ with $x \sqcup y \parallel x' \sqcup y', x \sqcup z \parallel x' \sqcup z', y \sqcup z \parallel y' \sqcup z'$
- (Pap) $\forall u, x, y, z, x' \in X$, $\{u, x, x'\}_{\neq}$ with $u \sqcup x = u \sqcup y = u \sqcup z \to \exists y', z' \in X$ with $u \sqcup x' = u \sqcup y' = u \sqcup z'$ and $x \sqcup x' \parallel z \sqcup z', x \sqcup y' \parallel y \sqcup z', y \sqcup x' \parallel z \sqcup y'$.

If a line L takes the form $x \sqcup y$ then the point x is called the basepoint of the line L. It is a simple consequence of the axioms that any line has either exactly one basepoint or all its points are basepoints (cf. [9]). A line all of its points are basepoints is called straight line. A line which is not straight (and hence has exactly one basepoint) is called a proper line.
A group $\mathbf{G}$ acting on a set X is called normally transitive if $\mathbf{G}$ is transitive and

$\mathbf{G}_x \setminus \mathbf{G}_y \neq \emptyset$ hold for any $x, y \in X$ with $x \neq y$ ($\mathbf{G}_x$ denotes the stabilizer of a point x with respect to $\mathbf{G}$). For any group acting on a set X one can construct a group space $\mathbf{V}(\mathbf{G}) = (X, \sqcup, \|)$ with
$x \sqcup y = \mathbf{G}_x\{x, y\} = \{x\} \cup \mathbf{G}_x y$ and
$L \parallel L'$ if there exists $g \in \mathbf{G}$ such that $gL = L'$ for any lines L, L'.
The following theorem will be basic in our construction.

Theorem 1.1. *The group space* $\mathbf{V}(\mathbf{G})$ *with respect to a normally transitive group* $\mathbf{G}$ *is a desarguesian skewaffine space.*

More detailed discussion of properties of the group space $\mathbf{V}(\mathbf{G})$ can be found in [9].

2. Residual skewaffine plane

Let $\langle p, K\rangle$ be a fixed pencil of circles of Laguerre plane $\mathbb{L} = (\mathcal{P}, \mathcal{C}, -)$ and $\Delta(p, K)$ a group of automorphisms of $\mathbb{L}$ which fixes the pencil $\langle p, K\rangle$ (not pointwiese) and all the points parallel to p. We assume that $\Delta(p, K)$ satisfies the following conditions:

(A1) $\Delta(p, K)$ is transitive on the set $\mathcal{P} \setminus p$.
(A2) $\Delta(p, K)_r$ is circular transitive for any $r \in K \setminus \{p\}$ (i.e. for any $x, y \in K \setminus \{p, r\}$ there exists $\sigma \in \Delta(p, K)_r$ such that $\sigma(x) = y$)).

Definition 2.1. A *residual skewaffine plane with respect to* $\langle p, K\rangle$ (written $\mathbb{SA}(p, K)$) is the group space $\mathbf{V}(\Delta(p, K) = (\mathcal{P} \setminus \overline{p}, \sqcup, \|)$.

By the definition $\Delta(p, K)$ is normally transitive and according to Theorem 1([1], p. 5) (comp. also [9] Proposition 6.5, p. 94) we get

Theorem 2.1. *Suppose* $\langle p, K\rangle$ *is a fixed pencil of a Laguerre plane* $\mathbb{L} = (\mathcal{P}, \mathcal{C}, -)$ *and conditions* (A1), (A2) *are satisfied. Then the residual skewaffine plane* $\mathbb{SA}(p, K)$ *is a skewaffine desarguesian space.*

In the following let $\langle p, K\rangle$ be a fixed pencil such that the corresponding group $\Delta(p, K)$ satisfies the conditions (A1), (A2).

Proposition 2.1. *a) If* $r \not- x$, *then* $r \sqcup x = M \setminus \overline{p}$, *where* M *is a circle of* $\langle p, K\rangle$ *or a circle tangent in the point* r *to some circle of* $\langle p, K\rangle$.
b) If $r - x$, *then* $r \sqcup x \subseteq \overline{r}$.

Proof. a) If $r \notin K$, then consider a circle $L = (p, K, r)^\circ$. First assume $x \in L$. If r' is an arbitrary point of K different from p there exists an automorphism $\alpha \in \Delta(p, K)$ such that $\alpha(r) = r'$. From the definition of the group $\Delta(p, K)$, by the touching axiom we get $\alpha(L) = K$. Hence, by (A2), $K \setminus \{p\} = \Delta(p, K)_{r'}\alpha(x)$ and $L \setminus \{p\} = \alpha^{-1}(K \setminus \{p\}) = \Delta(p, K)_r x$. In the case $x \notin L$ we define $M = (r, L, x)^\circ$, $x' = xL$ and $y \in M \setminus \overline{p}$, $y \neq r$. Similarly as above we conclude that M is invariant with respect to the group $\Delta(p, K)_r$. By the proved part of the proposition there exists $\beta \in \Delta(p, K)_r$ such that $\beta(x') = yL$. Hence $\beta(x) = y$.
b) It follows directly from the definition. □

Definition 2.2. The line $x \sqcup y$ of $\mathbb{SA}(p, K)$ is called *special* if $x - y$.

From the proof of Proposition 2.1 we get the generalization of the property (A2).

Corollary 2.1. *Let* $r \notin \overline{p}$ *and* M *is invariant with respect to* $\Delta(p, K)_r$. *Then for any* $x, y \in M$ *with* $x, y \neq pM, r$ *there exists* $\sigma \in \Delta(p, K)_r$ *such that* $\sigma(x) = y$.

Proposition 2.2. *The lines determined by the circles of the pencil $\langle p, K\rangle$ are straight lines.*

Proof. If x, y are distinct points of a circle $L \in \langle p, K\rangle$, then $\Delta(p, K)_x(y) = L \setminus \{p\}$ by Corollary 2.1. □

The group $\Delta(p, K)$ fixes the generator $\overline{p}$ pointwiese. Hence by the definition of parallelity we get:

Proposition 2.3. *For any $M, N \in \mathcal{C}$ if $M \setminus \overline{p} \parallel L \setminus \overline{p}$, then $pM = pL$.*

Propositions 2.1, 2.2, 2.3 provide the representation of basic notions of residual skewaffine plane of Laguerre plane. However this representation is not complete. The construction does not assume that any circle of the Laguerre plane corresponds to some line of skewaffine plane. Similarly, the statements inverse to Propositions 2.2 and 2.3 do not hold (as an example we can take any miquelian Laguerre plane of the characteristic 2). To obtain this representation more complete the following assumption will be needed throughout the further of the paper.
(A3) For any circle M such that $p \notin M$ there exists exactly one circle $L \in \langle p, K\rangle$ tangent to M.

Proposition 2.4. *If M is a circle such that $p \notin M$ then $M \setminus \overline{p} = x \sqcup y$, where x is the point of tangency M with the unique circle of the pencil $\langle p, K\rangle$ and y is any point of the circle M different from x and pM.*

Proof. According to (A3) there exists exactly one circle $L \in \langle p, K\rangle$ tangent to M. Let x be its point of tangency with M. The circle L is invariant with respect to $\Delta(p, K)_x$, since $L \in \langle p, K\rangle$ and $x \neq p$ is fixed. Hence M, as a circle tangent to an invariant circle and containing a fixed point pM is invariant with respect to $\Delta(p, K)_x$. The assertion follows from Corollary 2.1. □

According to Proposition 2.4 in the case $x \sqcup y = M \setminus \overline{p}$ the base point x of the line $x \sqcup y$ we will also call the *base point* of the circle M.

Proposition 2.5. *If $M \setminus \overline{p}$ is a straight line, then $M \in \langle p, K\rangle$.*

Proof. Assume that $M \setminus \overline{p} = x \sqcup y = y \sqcup x$ for some distinct points $x, y \in M \setminus \overline{p}$ and $M \notin \langle p, K\rangle$. Then $(x, M, p)^\circ$ and $(y, M, p)^\circ$ would be two distinct circles of the pencil $\langle p, K\rangle$ tangent to M, a contradiction with (A3). □

Proposition 2.6. *If $pM = pL$ with $p \notin M, L$, then $M \setminus \overline{p} \parallel L \setminus \overline{p}$.*

Proof. From the Proposition 2.4 $M \setminus \overline{p} = x \sqcup y$ where x is a point of tangency M with some $M' \in \langle p, K\rangle$ and $L \setminus \overline{p} = z \sqcup t$ where z is a point of tangency L with some $L' \in \langle p, K\rangle$. By (A1) there exists $\sigma \in \Delta(p, K)$ such that $\sigma(x) = z$. We obtain $\sigma(M') = L'$ and hence $\sigma(M) = L$ by the touching axiom. □

Remark 2.1. In the case of miquelian planes of the characteristic different from 2 any pencil $\langle p, K\rangle$ satisfies (A3) and the group $\Delta(p, K)$ has properties (A1) and (A2). Additionally, $\mathbf{V}(\Delta(p, K))$ fulfills (Papp) and (Pgm).

Remark 2.2. Examples of nonmiquelian (even nonovoidal) planes satisfying (A1), (A2), (A3) one may obtain if we put in the construction from [2] $f(x) = |x|^r$ $(r > 1)$ and $p = (\infty, 0)$, $K = \{(x, 0)|x \in K\} \cup \{(\infty, 0)\}$. Transformations of the form $x' = kx$; $y' = |k|^r y$ (identity for points (∞, a)) form the stabilizer of a point $(0, 0)$.

Remark 2.3. Axiom (A3) is satisfied for any pencil $\langle p, K\rangle$ of a topological Laguerre plane of dimension 2 and 4 (it is a special case of the solution of Apolonius problem for such planes cf. [10]).

3. A characterization of the group $\Delta(p, K)$.

Theorem 3.1. *For any point $r \notin \overline{p}$ the stabilizer $\Delta(p, K)_r$ is a $(\overline{p}, \langle r, L\rangle)$-transitive group of $\mathbb{L}$-strains $\Delta(\overline{p}, r, L)$ containing $\mathrm{S}_{\overline{r},\overline{p};K}$, where $L = (p, K, r)^\circ$.*

Proof. For any $\phi \in \Delta(p, K)_r$, $\phi(L) = L$, since $\Delta(p, K)$ fixes $\langle p, K\rangle$. Hence $\phi(M) = M$ for all $M \in \langle r, L\rangle$, because $\overline{p}$ is poinwiese fixed. The $(\overline{p}, \langle r, L\rangle)$-transitivity of the group $\Delta(p, K)$ follows from Corollary 2.1. To obtain $\mathrm{S}_{\overline{r},\overline{p};K}$ let us consider an arbitrary $x \notin L\cup\overline{r}\cup\overline{p}$ and circles $M = (r, L, x)^\circ$, $N = (p, L, x)^\circ$. According to (A3) circles M and N are not tangent, so there exists a point y with $y \neq x$, $y \in M \cap N$. By $(\overline{p}, \langle r, L\rangle)$-transitivity there exist $\psi \in \Delta(p, K)_r$ such that $\psi(x) = y$. Hence $\psi(N) = N$ and $\psi(y) = x$. It shows that $\psi = \mathrm{S}_{\overline{r},\overline{p};K}$. □

Proposition 3.1. *The group $\Delta(p, K)$ is $\langle p, K\rangle$-transitive.*

Proof. Let $x, y \in K$ be such that $\{x, y, p\}_{\neq}$. Consider an arbitrary circle M such that $M \cap K = \{x, y\}$ and the base point r of M. According to Theorem 3.1 there exist the symmetries $\mathrm{S}_{\overline{r},\overline{p};K}$ and $\mathrm{S}_{\overline{x},\overline{p};K}$ and the superposition $\mathrm{S}_{\overline{r},\overline{p};K} \circ \mathrm{S}_{\overline{x},\overline{p};K}$ is a translation which transforms x on y. □

From the proof of Proposition 3.1 we obtain:

Corollary 3.1. *For any distinct points x, y such that $x, y \neq p$, $x, y \in R \in \langle p, K\rangle$ there exists $r \in R$ such that $\mathrm{S}_{\overline{r},\overline{p};K}(x) = y$*

Lemma 3.1. *Any fixpoint free (outside $\overline{p}$) automorphism from $\Delta(p, K)$ is a translation.*

Proof. Suppose there exists x such that $x \not\parallel \phi(x)$ for some ϕ fulfilling the assumptions of the Lemma. Then $\phi(x) \not\parallel \phi^2(x)$. If $x \neq \phi^2(x)$, then the automorphism ϕ fixes the circle $M = (x, \phi(x), \phi^2(x))^\circ$. If $x = \phi^2(x)$, then ϕ fixes any circle $M \in \langle x, \phi(x)$. Hence $p \in M$ unless otherwise a base point of the circle M is fixed. It means that ϕ is a translation with an invariant pencil $\langle p, M\rangle$. In the case $x - \phi(x)$ the assumption that there exists y such $y \not\parallel \phi(y)$ follows that the circle $N = (y, \phi(y), \phi^2(y))^\circ$ is an invariant and a point of intersection N with $\overline{x}$ is fixed, a contradiction. Thus in this case ϕ is a translation which fixes all generators. □

Proposition 3.2. *The group $\Delta(p, K)$ is $\overline{p}$-transitive.*

Proof. Let $x - y$ and $r \not\parallel x$. Suppose $\phi \in \Delta(p, K)_r$ an arbitrary, $z = \phi(x)$, $z' = \mathrm{S}_{\overline{p},\overline{r};K}(z)$ and $M = (y, z, z')^\circ$. Because $\mathrm{S}_{\overline{p},\overline{r};K}(M) = M$ the base point s of the circle M is parallel to r. By Theorem 3.1 there exists $\psi \in \Delta(p, K)_s$ such that $\psi(z) = y$, so $\psi \circ \phi(x) = y$. The automorphism $\psi \circ \phi$ fixes (not pointwiese) two generators distinct from $\overline{p}$. Besides it is not an $\mathbb{L}$-strain so it is fixpoint free outside $\overline{p}$. By Lemma 3.1 $\psi \circ \phi$ is a translation which maps x on y. □

The group of translations contained in $\Delta(p, K)$ will be denoted by $\mathbf{T}(p, \Delta)$.

Theorem 3.2. *Elements of the group $\Delta(p, K)$ without fixed points (outside $\overline{p}$) are translations in direction of any L ($p \in L$) and translations which fix generators. The group $\mathbf{T}(p, \Delta)$ is transitive on the set $\mathcal{P} \setminus \overline{p}$ and $\mathbf{T}(p, \Delta) \trianglelefteq \Delta(p, K)$. Elements*

of $\Delta(p,K)$ with fixed points are $\mathbb{L}$-strains. The group $\Delta(p,K)$ is isomorphic to $\simeq \mathbf{T}(p,\Delta) \rtimes \Delta(p,K)_r$ for any $r \notin \overline{p}$.

Proof. The first part of the theorem follows from Proposition 3.1, Proposition 3.2, and [7], Theorem 4.19. For any $\mathbb{L}$-strain ϕ and a translation ψ the superposition $\phi \circ \psi \circ \phi^{-1}$ is a translation. Indeed, otherwise $x = \phi \circ \psi \circ \phi^{-1}(x)$ for some $x \in \overline{p}$ and $\phi^{-1}(x)$ is the fixed point of the translation ψ, a contradiction. □

Corollary 3.2. *The group $\Delta(p,K)$ is of type* **1H** *in the classification* [8] *of Laguerre planes.*

Corollary 3.3. *The group space $\mathbf{V}(\Delta(p,K)$ satisfies the condition* (Pgm).

Proof. The group $\mathbf{T}(p,\Delta)$ is commutative ([7], Theorem 4.14) and transitive. Hence the condition 5 is satisfied ([7], Proposition 6.5). □

Corollary 3.4. *Lines A, B of the residual skewaffine plane are parallel iff there exists a translation ϕ such that $\phi(A) = B$.*

4. Some properties of residual skewaffine plane and their applications to Laguerre plane

Proposition 4.1. *There are no three circles L, M, N that are tangent at different points with $L \in (p,K)$ and $M \cap N \subset \overline{p}$.*

Proof. Suppose circles M, N are tangent to the circle L of the pencil $\langle p, K\rangle$ in points x, y respectively and M, N have common point on the generator $\overline{p}$. By Corollary 3.1 there exists $r \in L \setminus \overline{p}$ such that $\mathrm{S}_{\overline{r},\overline{p};K}(x) = y$. We obtain $\mathrm{S}_{\overline{r},\overline{p};K}(M) = N$ and hence $rM = rN$ is another common point of M, N. □

Corollary 4.1. *Parallel lines of $\mathbb{SA}(p,K)$ determined by circles with base points on a straight line determined by a circle have a common point.*

Proposition 4.2. *Proper parallel lines of $\mathbb{SA}(p,K)$ determined by circles are disjoint iff their base points are distinct and parallel.*

Proof. $\Leftarrow$ Let m, n be distinct and parallel base points of circles M, N and M, N have a common point on the generator $\overline{p}$. A translation $\alpha \in \mathbf{T}(p,\mathcal{G})$ such that $\alpha(m) = n$ transforms M on N, hence $(M \setminus \overline{p}) \cap (N \setminus \overline{p}) = \emptyset$.
$\Rightarrow$ Assume the circles M, N are tangent in a point $q \in \overline{p}$ and their base points m, n are not parallel. Denote $L = (p,K,m)^\circ$, $z = nL$ and $M' = (z,L,q)^\circ$. By Corollary 4.1 M' is not tangent to M. This is a contradiction with the proved part of the proposition. □

Proposition 4.3. *Suppose that base points of parallel lines A, A' belongs to a straight line B determined by a circle. If a straight line C determined by a circle intersects the line A then it intersects the line A'.*

Proof. Let x, x' be base points of lines A and A', respectively. Let also y be one of the common points of the lines A, C. There exists a translation $\tau \in \mathbf{T}(p,K)$ such that $\tau(x) = x'$. We obtain $\tau(A) = A'$ and the point $y' = \tau(y)$ is a common point of the lines A', C. □

Lemma 4.1. *Let circles P, Q, R are tangent to a circle $L \in \langle p, K\rangle$ in pairwiese distinct points. If Q have common points with P, R, then P, R have a common point.*

Proof. If two of the circles P, Q, R determine parallel lines, the assertion follows from Proposition 4.3. Assume $x \in P \cap Q$, $y \in Q \cap R$ and $x, y \notin \overline{p}$. If $Q \in \langle p, K\rangle$, consider a translation $\tau \in \mathrm{T}(p, K)$ such that $\tau(x) = y$ and the circle $P' = \tau(P)$. We obtain $y \in P' \cap R$ and P' is tangent to L. Hence, by Veblen-condition (V),the circles P, R have a common point. In the case $Q \cap L = \{r\} \neq \{p\}$ instead the translation τ we use the $\mathbb{L}$-strain $\phi \in \Delta(\overline{p}, K, r)$ such that $\phi(x) = y$ and the assertion follows by Veblen condition or Proposition 4.3. □

Definition 4.1. Let L be a circle L of the pencil $\langle p, K\rangle$. We say that points $a, b \notin L$ are *equivalent* (under L) and write $a \equiv_L b$ if $|P \cap Q| \geq 1$ for any circles P, Q tangent to L and passing through a and b, respectively.

Lemma 4.1 gives the following.

Proposition 4.4. *For any circle $L \in \langle p, K\rangle$ the relation $\equiv_L$ is an equivalence on the set $\mathcal{P} \setminus L$.*

Proposition 4.5. *For any points $a, b \in \mathcal{P} \setminus L$, $a \equiv_L b$ iff there exist circles P, Q tangent to L in distinct point and passing through a, b respectively such that $|P \cap Q| \geq 1$.*

The points set of a special line can be described by the relation $\equiv$ as follows:

Proposition 4.6. *Let $x \sqcup y$ be a special line, and let $L = (p, K, x)^\circ$, $z - y$, $z \neq x$. Then $z \in x \sqcup y$ iff $z \equiv_L y$.*

Proof. $\Rightarrow$ Assume $y' \in (p, K, y)^\circ$, $M = (x, L, y')^\circ$ for some $y' \neq p, y$. By the definition of $x \sqcup y$ there exists $\sigma \in \Delta(p, K)_x$ such that $\sigma(y) = z$. Then $\sigma(M) = M$. If $z' = \sigma(y')$, we obtain $y \equiv_L y' \equiv_L z' \equiv_L z$.
$\Leftarrow$ Let $y \in N \in \langle p, K\rangle$ and M be a circle passing through z tangent to L in a point different from p. By $z \equiv_L y$ there exists a point r such that $r \in M \cap N$. Denote $P = (x, L, r)^\circ$ and $Q = (p, K, z)^\circ$. Then P and Q have a common point s because $r \equiv_L z$. A strain $\phi \in \Delta(\overline{p}, K, x)$ such that $\phi(r) = s$ transforms y on z. □

Proposition 4.7. *If $x - p$, $x \neq p$, then for any point $y \notin L$, $x \equiv_L y$ iff there exist exactly two circles M, M' tangent to L such that $x, y \in M, M'$.*

Proof. It is sufficient to prove $\Rightarrow$. Let N be any circle tangent to L such that $x \in N$ and $P = (p, L, y)^\circ$. From $x \equiv_L y$ it follows that there exists $z \in P \cap N$. Then $M = \tau(N)$ where $\tau \in \mathrm{T}(p, K)$, $\tau(z) = y$. Then we obtain $M' = S_{\overline{p}, \overline{y}, K}(M)$. Suppose, contrary to our claim that there exists a circle M'' through x, y, tangent to L and distinct from M, M'. Denote r, r', r'' the base points of circles M, M', M'' respectively. There exists $\phi \in \Delta(p, K)_{r''}$ with $\phi(r) = r'$. We have: $\phi(M'') = M''$, $\phi(M) = M'$ and $\phi(y) \neq x, y$. Hence $x, y, \phi(y)$ are three distinct points of two distinct circles M', M'', a contradiction. □

Lemma 4.2. *For any q parallel to p and different from p there exists exactly one q' parallel to p with the property:*
$\forall x, y((x \not- y, p \not- x, y) \wedge q \in (x(p, K, x)^\circ, y)^\circ \rightarrow q' \in (y, (p, K, y)^\circ, x)^\circ)$.

Proof. The assertion is a consequence of the symmetry condition (P2). If the point q determines the class of lines parallel to a line $x \sqcup y$, then q' determines the class of lines parallel to $y \sqcup x$. □

Theorem 4.1. *Let $q \neq p$, $q - p$, $x \not- p$. Then the points of tangency of circles of the pencil $\langle p, K\rangle$ with circles of the pencil $\langle x, q\rangle$ form a circle (without a point of the generator $\overline{p}$).*

Proof. Let $L = (p, K, x)^\circ$ and $M = (x, L, q)^\circ$. The point x is the point of tangency of circles L and M of pencils $\langle p, K\rangle$ and $\langle q, x\rangle$, respectively. Consider an arbitrary circle $N \in \langle q, x\rangle$, $N \neq M$. By the condition (A3), there exists exactly one circle $P \in \langle p, K\rangle$ tangent to N in some point y. The circle $Q = (x, L, y)^\circ$ is fixed by the group $\Delta(p, K)_x$. According to Corollary 2.1, any point of Q distinct from x and pQ is an image of the point y in some $\sigma \in \Delta(p, K)_x$. Hence it is a point of tangency of circles of the pencils $\langle p, K\rangle$ and $\langle pQ, x\rangle$, respectively. It follows that the circle Q satisfies the assertion of the theorem. □

Corollary 4.2. *The circle Q determined in Theorem 4.1 pass through the point q' from Lemma 4.2*

In the case of miquelian Laguerre planes of the characteristic different from 2, the point p in Theorem 4.1 can be chosen arbitrarily by Remark 2.1. For such planes we also obtain the condition deciding whether a circle through two points tangent to a circle can be constructed.

Theorem 4.2. *Let x, y be points and L a circle with $x \not- y$, $x, y \notin L$ of miquelian Laguerre plane of characteristic distinct from 2. The following conditions are equivalent:*

1. *There exist exactly two circles through x, y tangent to L.*
2. *Any circle through x tangent to L intersects any circle through y tangent to L.*
3. *There exist two intersecting circles tangent to L in distinct points containing x, y, respectively.*

Proof. According to Remark 2.1 the assertion follows by Definition 4.1, Proposition 4.5 and Proposition 4.7 applied to the pencil $\langle xL, L\rangle$. □

Remark 4.1. In miquelian Laguerre planes over a field $\mathbb{F}$ of characteristic different from 2 the conditions of Theorem 4.2 define the relation "$\equiv_L$" for any circle L. In an analytic representation of such planes for a circle $K = \{(x, 0) | x \in \mathrm{F}\} \cup \{(\infty, 0)\}$ points (a_1, b_1) and (a_2, b_2) are equivalent with respect to K iff $b_2 \in b_1\mathrm{F}^2$. In this case classes of parallelity of special lines corresponds to classes of squares of $\mathbb{F}$.

Remark 4.2. If $\mathbb{F}$ is quadratically closed, then any special line coincides with a generator and is a straight line. In this case $\mathbb{SA}(p, K)$ contains two families of straight lines as nearaffine residual plane connected with Minkowski plane (cf. [11]). But the class of straight lines determined by the circles of the pencil $\langle p, K\rangle$ do not satisfy the condition about existence of exactly one common point with other lines.

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Andrzej Matraś, Department of Mathematics and Informatics, UWM Olsztyn, Żołnierska 14, 10-561 Olsztyn, Poland
E-mail address: `matras@uwm.edu.pl`